A COUNTABLE FREE CLOSED NON-REFLEXIVE SUBGROUP OF $\mathbb{Z}^{\mathfrak{c}}$

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ABSTRACT. We prove that the group $G = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ of all homomorphisms from the Baer-Specker group $\mathbb{Z}^{\mathbb{N}}$ to the group $\mathbb{Z}$ of integer numbers endowed with the topology of pointwise convergence contains no infinite compact subsets. From this it follows that the second Pontryagin dual of G is discrete. Since G is non-discrete, it is not reflexive. Since G can be viewed as a closed subgroup of the Tychonoff product $\mathbb{Z}^{\mathfrak{c}}$ of continuum many copies of the integers $\mathbb{Z}$, this provides an example of a group described in the title, thereby answering Problem 11 from [J. Galindo, L. Recorder-Núñez, M. Tkachenko, *Reflexivity of prodiscrete topological groups*, J. Math. Anal. Appl. 384 (2011), 320–330].

1. Introduction

The circle group $\mathbb{T}$ is the quotient of the real numbers over the subgroup of integer numbers. (Alternatively, one can view $\mathbb{T}$ as the unit circle in the complex plane with multiplication as a group operation.) A *character* of a topological group G is a continuous group homomorphism from G to the circle group $\mathbb{T}$. The set of all characters on an abelian topological group G forms an abelian group $\widehat{G}$ with pointwise addition as its group operation. When equipped with the compact-open topology, $\widehat{G}$ becomes a topological group called the (Pontryagin) *dual group* of G.

We say that an abelian topological group G is *reflexive* provided that the second Pontryagin dual $\widehat{\widehat{G}}$ of G is topologically isomorphic to G. The celebrated theorem of Pontryagin-van Kampen says that all locally compact abelian groups are reflexive. Understanding the duality features of large classes of topological abelian groups beyond locally compact ones is a major topic of the topological group theory. Extending the scope of the Pontryagin-van Kampen theorem, Banasczyk [3] introduced the class of nuclear groups and investigated its duality properties.

The class of nuclear groups contains all locally compact abelian groups and is closed under taking products, subgroups and quotients [3, 7.5 and 7.10]. It follows from this that every prodiscrete abelian group is nuclear. (Recall that a topological group is *prodiscrete* if it is a closed subgroup of some product of discrete groups.)

Hofmann and Morris [9, Problem 5.3] propose to develop both a structure and a character theory of prodiscrete groups. (Recently, they re-phrased this problem in [10, Question 1] asking if there exists a satisfactory structure theory even for abelian non-discrete prodiscrete groups.) In item (a) of [9, Problem 5.3] the special case of compact-free prodiscrete groups is considered, and as a typical example, the group $G = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ is proposed, and it is explicitly mentioned that the dual and bi-dual of this group has not been investigated. We show that the bi-dual $\widehat{\widehat{G}}$ of G is the group itself endowed with the discrete topology. Since G is non-discrete, it follows that G is non-reflexive.

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Galindo, Recorder-Núñez and Tkachenko [8] found many examples of non-reflexive prodiscrete groups. Indeed, they show that any bounded torsion abelian group of cardinality $\mathfrak{c}$ admits a Hausdorff group topology making it into a prodiscrete non-reflexive group of countable pseudocharacter. (Recall that a topological group G has *countable pseudocharacter* if its identity is an intersection of a countable family of open subsets of G.) In [8, Problem 11], Galindo, Recorder-Núñez and Tkachenko ask if $\mathbb{Z}^{\mathfrak{c}}$ contains a closed subgroup of countable pseudocharacter which is not reflexive.

Note that $G = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ can be viewed as a closed subgroup of $\mathbb{Z}^{\mathfrak{c}}$. Furthermore, it is well known that G is algebraically isomorphic to the free abelian group with countably many generators (see Fact 2.2); in particular, G is countable. Since countable groups have countable pseudocharacter, our example gives a strong positive answer to the problem of Galindo, Recorder-Núñez and Tkachenko.

Banasczyk [3, Remark 17.14] says that it remains an open question whether closed subgroups or Hausdorff quotients of uncountable products of $\mathbb{R}$'s and $\mathbb{Z}$'s are reflexive. Answering half of this question, Außenhofer [2] found a non-reflexive Hausdorff quotient of $\mathbb{Z}^{\mathfrak{c}}$. We answer another half of this question by noticing that $G = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ is a (countable) closed non-reflexive subgroup of $\mathbb{Z}^{\mathfrak{c}}$.

Negrepontis [12] proved that direct and inverse limits of compactly generated LCA groups are reflexive. Banasczyk [3, Remark 17.14] points out an error in his proof. Finally, our counter-example shows that the statement of the inverse limit theorem from [12] is also wrong. Indeed, our closed non-reflexive subgroup G of $\mathbb{Z}^{\mathfrak{c}}$ is an inverse limit of discrete (thus, locally compact) finitely generated free abelian groups.

2. The Baer-Specker group and the group of its integer-valued homomorphisms

We denote by $\mathbb{Z}$ the group of integer numbers equipped with the discrete topology. The group

$$X = \mathbb{Z}^{\mathbb{N}}$$

is called the *Baer-Specker group*. We consider the Baer-Specker group $X = \mathbb{Z}^{\mathbb{N}}$ with the Tychonoff product topology.

For abelian groups A and B we denote by $\mathrm{Hom}(A, B)$ the group of all homomorphisms from A to B. Let

$$G = \mathrm{Hom}(X, \mathbb{Z})$$

be the group of all group homomorphisms from the Baer-Specker group $\mathbb{Z}^{\mathbb{N}}$ to the group $\mathbb{Z}$ of integer numbers. Clearly, G can be viewed as a subgroup of the direct product $\mathbb{Z}^X$. One easily checks that G is a closed subgroup of $\mathbb{Z}^X$ when the latter group is equipped with the Tychonoff product topology. This topology induces on G the *topology of pointwise convergence* which we consider in this paper.

Fact 2.1. *Let $g \in G = \mathrm{Hom}(X, \mathbb{Z})$. Then:*

(i) *there exist an integer $n \in \mathbb{N}$ and a homomorphism $g' \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that $g = g' \circ \pi_n$, where $\pi_n : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^n$ is the projection on the first n coordinates;*

(ii) *g is continuous.*

Proof. The proof of item (i) can be found in [15].

Item (ii) easily follows from item (i). Indeed, let n and g' be as in item (i). Since $\mathbb{Z}^n$ is discrete, g' is continuous. Since π_n is also continuous, the continuity of g follows.

Another proof of item (ii) can be found in [13]. □

The following fact is well known; see for example, 1.3 and 2.4 in Chapter III of [6]. It also follows easily from Fact 2.1 (i).

Fact 2.2. *For every $n \in \mathbb{N}$, let $g_n : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ denote the projection on nth coordinate defined by $g_n(x) = x(n)$ for $x \in \mathbb{Z}^{\mathbb{N}}$. Then $\{g_n : n \in \mathbb{N}\}$ is an independent set of generators of G; in particular, G is the free abelian group with countably many generators.*

The next proposition is probably known. We present its proof for the reader's convenience only.

Proposition 2.3. *G is not discrete.*

Proof. Let Y be a finite subset of X. Consider a basic open neighbourhood

$$U_Y = \{g \in G : g(y) = 0 \text{ for all } y \in Y\} \tag{1}$$

of 0 in G. It suffices to show that $U_Y \neq \{0\}$. By the well-known property of the Baer-Specker group, there exists a finitely generated subgroup H of X containing Y and a subgroup N of X such that $X = H \oplus N$; see, for example, [7, proof of (19.2)]. Since X is not finitely generated, $N \neq \{0\}$. Therefore, there exists a non-trivial homomorphism $f : N \to \mathbb{Z}$. (One can take as f the restriction to N of a suitable projection from $X = \mathbb{Z}^{\mathbb{N}}$ to $\mathbb{Z}$.) Now let $g \in G$ be such that that $g\restriction_H = 0$ and $g\restriction_N = f$. Since $Y \subseteq H$, one has $g \in U_Y$. On the other hand, $g \neq 0$ as $g\restriction_N = f \neq 0$. □

3. Results

Our principal result is the following

Theorem 3.1. *All compact subsets of the group $G = \mathrm{Hom}(X, \mathbb{Z})$ are finite.*

Theorem 3.1 enables us to compute the second dual $\widehat{\widehat{G}}$ of G.

Corollary 3.2. *$\widehat{\widehat{G}}$ is discrete and algebraically isomorphic to G.*

Proof. Since compact subsets of G are finite by Theorem 3.1, the dual group $\widehat{G}$ has the topology of pointwise convergence on G; that is, it can be identified with the subgroup of $\mathbb{T}^G$ endowed with the Tychonoff product topology. Since G is countable, the dual group $\widehat{G}$ of G is (precompact and) metrizable. Therefore, the second dual $\widehat{\widehat{G}}$ of G is a k-space; see [4, 1].

Since G is prodiscrete, the natural map $\alpha_G : G \to \widehat{\widehat{G}}$ from G to its second dual $\widehat{\widehat{G}}$ is both open and an algebraic isomorphism between G and $\widehat{\widehat{G}}$; see [8, Theorem 3.1(2)]. Therefore, its inverse is a continuous isomorphism from $\widehat{\widehat{G}}$ onto G. Since all compact subsets of G are finite by Theorem 3.1, the same holds for $\widehat{\widehat{G}}$. Since $\widehat{\widehat{G}}$ is a k-space, it must be discrete. □

Proposition 2.3 and Corollary 3.2 imply the following

Corollary 3.3. *G is non-reflexive.*

From Fact 2.2 and this corollary we get the following

Corollary 3.4. *G is a non-reflexive closed free subgroup of $\mathbb{Z}^X$ with countably many generators.*

Since $|X| = |\mathbb{Z}^{\mathbb{N}}| = \mathfrak{c}$, the topological groups $\mathbb{Z}^X$ and $\mathbb{Z}^{\mathfrak{c}}$ are topologically isomorphic. Thus, we obtain the following

Corollary 3.5. *$\mathbb{Z}^{\mathfrak{c}}$ contains a countable closed non-reflexive subgroup.*

In [8, Problem 11], Galindo, Recorder-Núñez and Tkachenko ask if $\mathbb{Z}^{\mathfrak{c}}$ contains a closed subgroup of countable pseudocharacter which is not reflexive. Since countable topological groups have countable pseudocharacter, this corollary provides a strong positive answer to this problem. Earlier, Pol and Smentek [14, 6.3] constructed a countable closed non-reflexive subgroup in the power $A^{\mathfrak{c}}$ of the free abelian group A with countably many generators equipped with the discrete topology.

Recall that a topological abelian group G is called *strongly reflexive* if all closed subgroups and all quotient groups of G are reflexive; see, for example, [5].

Corollary 3.5 implies the main result of Außenhofer from [2]:

Corollary 3.6. ([2, Corollary 3.6]) *$\mathbb{Z}^{\mathfrak{c}}$ is not strongly reflexive.*

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is split into a sequence of claims.

Claim 1. For every $x \in X$, the map $\varphi_x \in \mathrm{Hom}(G, \mathbb{Z})$ defined by $\varphi_x(g) = g(x)$ for all $g \in G$, is continuous.

Claim 2. The evaluation map $e : X \times G \to \mathbb{Z}$ defined by $e(x, g) = g(x)$ for $x \in X$ and $g \in G$, is separately continuous.

Proof. For a fixed $x \in X$, the map $g \mapsto g(x)$ coincides with φ_x, so it is continuous by Claim 1.

For a fixed $g \in G$, the map $x \mapsto g(x)$ coincides with g, so its continuity follows from Fact 2.1 (ii). □

Claim 3. For every non-empty compact subset K of G, there exists $m \in \mathbb{N}$ (depending on K) such that $g(O_m) \subseteq \{0\}$ for all $g \in K$, where

$$O_m = \{x \in X : x(i) = 0 \text{ for all } i = 1, 2, \ldots, m\}. \tag{2}$$

Proof. Let K be a non-empty compact subset of G. Let $\varepsilon : X \times K \to \mathbb{Z}$ be the restriction of the evaluation map e to the subset $X \times K$ of $X \times G$. Then ε is separately continuous by Claim 2. Since $X = \mathbb{Z}^{\mathbb{N}}$ is a complete separable metric space and K is compact, we can apply Namioka's theorem [11] to find a dense G_δ-subset D of X such that ε is jointly continuous at every point of the set $D \times K$. Since $D \neq \emptyset$, we can choose $x_0 \in D$.

For every $g \in K$, we can use the continuity of ε at (x_0, g) and the discreteness of $\mathbb{Z}$ to fix an open neighbourhood U_g of x_0 in X and an open neighbourhood V_g of g in K such that $\varepsilon(U_g \times V_g) = \{\varepsilon(x_0, g)\}$. Since K is compact, there exist $g_1, g_2, \ldots, g_k \in G$ such that $K \subseteq \bigcup_{i=1}^k V_{g_i}$. Now $U = \bigcap_{i=1}^k U_{g_i}$ is an open neighbourhood of x_0 in X such that

$$g(x) = \varepsilon(x, g) = \varepsilon(x_0, g) = g(x_0) \text{ for all } x \in U \text{ and each } g \in K. \tag{3}$$

Since U is an open neighbourhood of x_0 in $X = \mathbb{Z}^{\mathbb{N}}$, there exists $m \in \mathbb{N}$ such that

$$W = \{x \in X : x(i) = x_0(i) \text{ for all } i = 1, 2, \ldots, m\} \subseteq U. \tag{4}$$

Let $x \in O_m$ and $g \in K$ be arbitrary. Define $y_0 \in X$ by

$$y_0(i) = \begin{cases} x_0(i) & \text{if } i = 1, 2, \dots, m \\ 0 & \text{if } i > m \end{cases} \qquad \text{for all} \quad i \in \mathbb{N}.$$

Since $y_0 \in W$ and $x+y_0 \in W$, from (3) and (4) we obtain $g(y_0) = g(x_0)$ and $g(x+y_0) = g(x_0)$. Since g is a homomorphism, $g(x + y_0) = g(x) + g(y_0)$. This gives $g(x) = 0$. We have proved that $g(O_m) \subseteq \{0\}$ for all $g \in K$. □

For each $n \in \mathbb{N}$, let $\delta_n \in X$ be the function defined by

$$\delta_n(i) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases} \qquad \text{for all} \quad i \in \mathbb{N}. \tag{5}$$

Define

$$\psi_n = \varphi_{\delta_n} \in \mathrm{Hom}(G, \mathbb{Z}). \tag{6}$$

Claim 4. For every non-empty compact subset K of G, there exists $m \in \mathbb{N}$ (depending on K) such that $\psi_n(K) = \{0\}$ for all $n \in \mathbb{N}$ with $n > m$.

Proof. Let $m \in \mathbb{N}$ be as in the conclusion of Claim 3. If $n \in \mathbb{N}$ and $n > m$, then $\delta_n \in O_m$ by (2) and (5), so $\psi_n(g) = \varphi_{\delta_n}(g) = g(\delta_n) = 0$ for all $g \in K$; that is, $\psi_n(K) = \{0\}$. □

Let $\psi \in \mathrm{Hom}(G, \mathbb{Z}^{\mathbb{N}})$ be the diagonal product of the family $\{\psi_n : n \in \mathbb{N}\} \subseteq \mathrm{Hom}(G, \mathbb{Z})$ defined by $\psi(g)(n) = \psi_n(g)$ for all $n \in \mathbb{N}$ and $g \in G$; that is, the nth coordinate of $\psi(g) \in \mathbb{Z}^{\mathbb{N}}$ is $\psi_n(g)$.

Claim 5. ψ is continuous.

Proof. Each ψ_n is continuous by (6) and Claim 1. It remains only to recall that the diagonal product of continuous maps is continuous. □

Claim 6. ψ is a monomorphism.

Proof. Assume that $g \in G$ and $\psi(g) = 0$. Then $g(\delta_n) = \psi_n(g) = \psi(g)(n) = 0$ for all $n \in \mathbb{N}$. Since g is a homomorphism from X to $\mathbb{Z}$, this implies $g(Y) = \{0\}$, where Y is the subgroup of X generated by the set $\{\delta_n : n \in \mathbb{N}\}$. Since Y is dense in X and g is continuous by Fact 2.1 (ii), it follows that $g(X) = \{0\}$; that is, $g = 0$. We proved that ψ has a trivial kernel, so it is a monomorphism. □

Claim 7. All compact subsets of G are finite.

Proof. Let K be a non-empty compact subset of G. Choose m as in the conclusion of Claim 4. Then

$$\psi(K) \subseteq \mathbb{Z}^m \times \{0\} \times \{0\} \times \cdots \times \{0\} \times \dots,$$

so $\psi(K)$ is discrete. On the other hand, ψ is continuous by Claim 5, and so $\psi(K)$ is compact. Therefore, $\psi(K)$ is finite. Since ψ is a one-to-one map by Claim 6, it follows that K is finite as well. □

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